Levelling-Up **Basic Mathematics**

Logarithms

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The aim of this document is to provide a short, self assessment programme for students who wish to acquire a basic competence in the use of logarithms.

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1. Logarithms (Introduction)

Let a and N be positive real numbers and let $N = a^n$. Then n is called the *logarithm of* N to the base a. We write this as

$$n = \log_a N.$$

Examples 1

(a) Since $16 = 2^4$, then $4 = \log_2 16$.

(b) Since $81 = 3^4$, then $4 = \log_3 81$.

(c) Since $3 = \sqrt{9} = 9^{\frac{1}{2}}$, then $1/2 = \log_9 3$.

(d) Since $3^{-1} = 1/3$, then $-1 = \log_3(1/3)$.

Exercise

Use the definition of logarithm given on the previous page to determine the value of x in each of the following.

1. $x = \log_3 27$
2. $x = \log_5 125$
3. $x = \log_2 (1/4)$
4. $2 = \log_x (16)$
5. $3 = \log_2 x$

2. Rules of Logarithms

Let a, M, N be positive real numbers and k be any number. Then the following important rules apply to logarithms.

$$
\begin{array}{llcl}
1. & \log_a MN & = & \log_a M + \log_a N \\
2. & \log_a \left(\frac{M}{N}\right) & = & \log_a M - \log_a N \\
3. & \log_a \left(m^k\right) & = & k \log_a M \\
4. & \log_a a & = & 1 \\
5. & \log_a 1 & = & 0
\end{array}
$$

3. Logarithm of a Product

1. ⟵ **Proof that** $\log_a MN = \log_a M + \log_a N$.

Examples 2

(a) $\log_6 4 + \log_6 9 = \log_6(4 \times 9) = \log_6 36$.

If $x = \log_6 36$, then $6^x = 36 = 6^2$.

Thus $\log_6 4 + \log_6 9 = 2$.

(b) $\log_5 20 + \log_4 \left(\frac{1}{4}\right) = \log_5 \left(20 \times \frac{1}{4}\right)$.

Now $20 \times \frac{1}{4} = 5$ so $\log_5 20 + \log_4 \left(\frac{1}{4}\right) = \log_5 5 = 1$.

Quiz. To which of the following numbers does the expression $\log_3 15 + \log_3 0 \cdot 6$ simplify?

(a) 4 (b) 3 (c) 2 (d) 1

4. Logarithm of a Quotient

1. $\longleftarrow$ **Proof that** $\log_a\left(\frac{M}{N}\right) = \log_a M - \log_a N$.

Examples 3

(a) $\log_2 40 - \log_2 5 = \log_2\left(\frac{40}{5}\right) = \log_2 8$.

If $x = \log_2 8$ then $2^x = 8 = 2^3$, so $x = 3$.

(b) If $\log_3 5 = 1.465$ then we can find $\log_3 0 \cdot 6$.

Since $3/5 = 0 \cdot 6$, then $\log_3 0 \cdot 6 = \log_3\left(\frac{3}{5}\right) = \log_3 3 - \log_3 5$.

Now $\log_3 3 = 1$, so that $\log_3 0 \cdot 6 = 1 - 1 \cdot 465 = -0 \cdot 465$

Quiz. To which of the following numbers does the expression $\log_2 12 - \log_2\left(\frac{3}{4}\right)$ simplify?

(a) 0 (b) 1 (c) 2 (d) 4

5. Logarithm of a Power

1. $\longleftarrow$ **Proof that** $\log_a\left(m^k\right) = k\log_a M$

Examples 4

(a) Find $\log_{10}(1/10000)$. We have $10000 = 10^4$, so $1/10000 = 1/10^4 = 10^{-4}$.

Thus $\log_{10}(1/10000) = \log_{10}\left(10^{-4}\right) = -4\log_{10} 10 = -4$, where we have used rule 4 to write $\log_{10} 10 = 1$.

(b) Find $\log_{36} 6$. We have $6 = \sqrt{36} = 36^{\frac{1}{2}}$.

Thus $\log_{36} 6 = \log_{36}\left(36^{\frac{1}{2}}\right) = \frac{1}{2}\log_{36} 36 = \frac{1}{2}$.

Quiz. If $\log_3 5 = 1 \cdot 465$, which of the following numbers is $\log_3 0 \cdot 04$?

(a) -2.930 (b) -1.465 (c) -3.465 (d) 2.930

6. Use of the Rules of Logarithms

In this section we look at some applications of the rules of logarithms.

Examples 5

(a) $\log_4 1 = 0$.

(b) $\log_{10} 10 = 1$.

(c) $$\log_{10} 125 + \log_{10} 8 = \log_{10}(125 \times 8) = \log_{10} 1000$$
$$= \log_{10}\left(10^3\right) = 3\log_{10} 10 = 3.$$

(d) $$2\log_{10} 5 + \log_{10} 4 = \log_{10}\left(5^2\right) + \log_{10} 4 = \log_{10}(25 \times 4)$$
$$= \log_{10} 100 = \log_{10}\left(10^2\right) = 2\log_{10} 10 = 2.$$

(e) $$3\log_a 4 + \log_a(1/4) - 4\log_a 2 = \log_a\left(4^3\right) + \log_a(1/4) - \log_a\left(2^4\right)$$
$$= \log_a\left(4^3 \times \tfrac{1}{4}\right) - \log_a\left(2^4\right) = \log_a\left(4^2\right) - \log_a\left(2^4\right)$$
$$= \log_a 16 - \log_a 16 = 0.$$

Exercise

Use the rules of logarithms to simplify each of the following.

1. $3\log_3 2 - \log_3 4 + \log_3\left(\frac{1}{2}\right)$

2. $3\log_{10} 5 + 5\log_{10} 2 - \log_{10} 4$

3. $2\log_a 6 - (\log_a 4 + 2\log_a 3)$

4. $5\log_3 6 - (2\log_3 4 + \log_3 18)$

5. $3\log_4(\sqrt{3}) - \frac{1}{2}\log_4 3 + 3\log_4 2 - \log_4 6$

7. Quiz on Logarithms

In each of the following, find x.

Begin Quiz

1. $\log_x 1024 = 2$

(a) 2^3 (b) 2^4 (c) 2^2 (d) 2^5

2. $x = (\log_a \sqrt{27} - \log_a \sqrt{8} - \log_a \sqrt{125})/(\log_a 6 - \log_a 20)$

(a) 1 (b) 3 (c) 3/2 (d) -2/3

3. $\log_c(10 + x) - \log_c x = \log_c 5)$

(a) 2.5 (b) 4.5 (c) 5.5 (d) 7.5

End Quiz

Score: Correct

8. Change of Bases

There is one other rule for logarithms which is extremely useful in practice. This relates logarithms in one base to logarithms in a different base. Most calculators will have, as standard, a facility for finding logarithms to the base 10 and also for logarithms to base e (natural logarithms). What happens if a logarithm to a different base, for example 2, is required? The following is the rule that is needed.

$$\boxed{\log_a c = \log_a b \times \log_b c}$$

1. ⟵ **Proof of the above rule**

The most frequently used form of the rule is obtained by rearranging the rule on the previous page. We have

$$\log_a c = \log_a b \times \log_b c \quad \text{so} \quad \log_b c = \frac{\log_a c}{\log_a b}\,.$$

Examples 6

(a) Using a calculator we find that $\log_{10} 3 = 0 \cdot 47712$ and $\log_{10} 7 = 0 \cdot 84510$. Using the above rule,

$$\log_3 7 = \frac{\log_{10} 7}{\log_{10} 3} = \frac{0 \cdot 84510}{0 \cdot 47712} = 1 \cdot 77124\,.$$

(b) We can do the same calculation using instead logs to base e. Using a calculator, $\log_e 3 = 1 \cdot 09861$ and $\log_e 7 = 1 \cdot 94591$. Thus

$$\log_3 7 = \frac{\ln 7}{\ln 3} = \frac{1 \cdot 94591}{1 \cdot 09861} = 1 \cdot 77125\,.$$

The calculations have all been done to five decimal places, which explains the slight difference in answers.

(c) Given only that $\log_{10} 5 = 0 \cdot 69897$ we can still find $\log_2 5$, as follows. First we have $2 = 10/5$ so

$$\begin{aligned}\log_{10} 2 &= \log_{10}\left(\frac{10}{5}\right)\\ &= \log_{10} 10 - \log_{10} 5\\ &= 1 - 0 \cdot 69897\\ &= 0 \cdot 30103\,.\end{aligned}$$

Then

$$\log_2 5 = \frac{\log_{10} 5}{\log_{10} 2} = \frac{0 \cdot 69897}{0 \cdot 30103} = 2 \cdot 32193\,.$$

Solutions to Quizzes

Solution to Quiz:

Using rule 1 we have

$$\log_3 15 + \log_3 0 \cdot 6 = \log_3(15 \times 0 \cdot 6) = \log_3 9$$

But $9 = 3^2$ so

$$\log_3 15 + \log_3 0 \cdot 6 = \log_3 3^2 = 2.$$

End Quiz

Solution to Quiz:

Using rule 2 we have

$$\log_2 12 - \log_2\left(\tfrac{3}{4}\right) = \log_2\left(12 \div \tfrac{3}{4}\right)$$

Now we have $12 \div \frac{3}{4} = 12 \times \frac{4}{3} = \frac{12\times 4}{3} = 16$.

Thus $\log_2 12 - \log_2\left(\frac{3}{4}\right) = \log_2 16 = \log_2 2^4$.

If $x = \log_2 2^4$, then $2^x = 2^4$, so $x = 4$. End Quiz

Solution to Quiz:

Note that

$$0 \cdot 04 = 4/100 = 1/25 = 1/5^2 = 5^{-2}.$$

Thus

$$\log_3 0 \cdot 04 = \log_3 \left(5^{-2}\right) = -2 \log_3 5.$$

Since $\log_3 5 = 1 \cdot 465$, we have

$$\log_3 0 \cdot 05 = -2 \times 1 \cdot 465 = -2.930.$$

End Quiz

Solutions to Problems

Problem 1.

Since

$$x = \log_3 27$$

then, by the definition of a logarithm, we have

$$3^x = 27.$$

But $27 = 3^3$, so we have

$$3^x = 27 = 3^3,$$

giving

$$x = 3.$$

$\square$

Problem 2.

Since $x = \log_{25} 5$ then, by the definition of a logarithm,

$$25^x = 5.$$

Now

$$5 = \sqrt{25} = 25^{\frac{1}{2}},$$

so that

$$25^x = 5 = 25^{\frac{1}{2}},$$

From this we see that $x = 1/2$. $\square$

Problem 3.

Since $x = \log_2(1/4)$, then, by the definition of a logarithm,

$$2^x = 1/4 = 1/(2^2) = 2^{-2}.$$

Thus $x = -2$. $\square$

Problem 4.

Since $2 = \log_x(16)$ then, by the definition of logarithm,

$$x^2 = 16 = 4^2.$$

Thus

$$x = 4.$$

$\square$

Problem 5.

Since $3 = \log_2 x$, by the definition of logarithm, we must have

$$2^3 = x.$$

Thus $x = 8$. $\square$

Problem 1.

Let $m = \log_a M$ and $n = \log_a N$, so, by definition, $M = a^m$ and $N = a^n$. Then

$$MN = a^m \times a^n = a^{m+n}\,,$$

where we have used the appropriate rule for exponents. From this, using the definition of a logarithm, we have

$$m + n = \log_a(MN).$$

But $m+n = \log_a M + \log_a N$, and the above equation may be written

$$\log_a M + \log_a N = \log_a(MN),$$

which is what we wanted to prove. $\square$

Problem 1.

As before, let $m = \log_a M$ and $n = \log_a N$. Then $M = a^m$ and $N = a^n$. Now we have

$$\frac{M}{N} = \frac{a^m}{a^n} = a^{m-n},$$

where we have used the appropriate rule for indices. By the definition of a logarithm, we have

$$m - n = \log_a \left(\frac{M}{N}\right).$$

From this we are able to deduce that

$$\log_a M - \log_a N = m - n = \log_a \left(\frac{M}{N}\right).$$

$\square$

Problem 1.

Let $m = \log_a M$, so $M = a^m$. Then

$$M^k = (a^m)^k = a^{mk} = a^{km},$$

where we have used the appropriate rule for indices. From this we have, by the definition of a logarithm,

$$km = \log_a \left(M^k\right).$$

But $m = \log_a M$, so the last equation can be written

$$k \log_a M = km = \log_a \left(M^k\right),$$

which is the result we wanted. $\square$

Problem 1. First of all, by rule 3, we have $3\log_3 2 = \log_3(2^3) = \log_3 8$. Thus the expression becomes

$$\log_3 8 - \log_3 4 + \log_3\left(\frac{1}{2}\right) = \left[\log_3 8 + \log_3\left(\frac{1}{2}\right)\right] - \log_3 4\,.$$

Using rule 1, the first expression in the [ ] brackets becomes

$$\log_3\left(8 \times \frac{1}{2}\right) = \log_3 4.$$

The expression then simplifies to

$$\log_3 4 - \log_3 4 = 0.$$

$\square$

Problem 2.

First we use rule 3:

$$3\log_{10} 5 = \log_{10}\left(5^3\right)$$

and

$$5\log_{10} 2 = \log_{10}\left(2^5\right).$$

Thus

$$3\log_{10} 5 + 5\log_{10} 2 = \log_{10}\left(5^3\right) + \log\left(2^5\right) = \log_{10}\left(5^3 \times 2^5\right),$$

where we have used rule 1 to obtain the right hand side. Thus

$$3\log_{10} 5 + 5\log_{10} 2 - \log_{10} 4 = \log_{10}\left(5^3 \times 2^5\right) - \log_{10} 4$$

and, using rule 2, this simplifies to

$$\log_{10}\left(\frac{5^3 \times 2^5}{4}\right) = \log_{10}\left(10^3\right) = 3\log_{10} 10 = 3.$$

$\square$

Problem 3.

Dealing first with the expression in brackets, we have

$$\log_a 4 + 2\log_a 3 = \log_a 4 + \log_a \left(3^2\right) = \log_a \left(4 \times 3^2\right),$$

where we have used, in succession, rules 3 and 2. Now

$$2\log_a 6 = \log_a \left(6^2\right)$$

so that, finally, we have

$$\begin{aligned} 2\log_a 6 - (\log_a 4 + 2\log_a 3) &= \log_a \left(6^2\right) - \log_a \left(4 \times 3^2\right) \\ &= \log_a \left(\frac{6^2}{4 \times 3^2}\right) \\ &= \log_a 1 \\ &= 0. \end{aligned}$$

□

Problem 4.

Dealing first with the expression in brackets we have

$$2\log_3 4 + \log_3 18 = \log_3\left(4^2\right) + \log_3 18 = \log_3\left(4^2 \times 18\right),$$

where we have used rule 3 first, and then rule 1. Now, using rule 3 on the first term, followed by rule 2, we obtain

$$\begin{aligned} 5\log_3 6 - (2\log_3 4 + \log_3 18) &= \log_3\left(6^5\right) - \log_3\left(4^2 \times 18\right) \\ &= \log_3\left(\frac{6^5}{4^2 \times 18}\right) \\ &= \log_3\left(\frac{2^5 \times 3^5}{4^2 \times 2 \times 9}\right) \\ &= \log_3\left(3^3\right) \\ &= 3\log_3 3 = 3, \end{aligned}$$

since $\log_3 3 = 1$.

$\square$

Problem 5.

The first thing we note is that $\sqrt{3}$ can be written as $3^{\frac{1}{2}}$. We first simplify some of the terms. They are

$$3\log_4\sqrt{3} = 3\log_4\left(3^{\frac{1}{2}}\right) = \frac{3}{2}\log_4 3,$$

$$\log_4 6 = \log_4(2\times 3) = \log_4 2 + \log_4 3.$$

Putting all of this together:

$$\begin{aligned}
&3\log_4(\sqrt{3}) - \tfrac{1}{2}\log_4 3 + 3\log_4 2 - \log_4 6 \\
&\quad= \frac{3}{2}\log_4 3 - \frac{1}{2}\log_4 3 + 3\log_4 2 - (\log_4 2 + \log_4 3) \\
&\quad= \left(\frac{3}{2} - \frac{1}{2} - 1\right)\log_4 3 + (3-1)\log_4 2 \\
&\quad= 2\log_4 2 = \log_4\left(2^2\right) = \log_4 4 = 1.
\end{aligned}$$

$\square$

Problem 1.

Let $x = \log_a b$ and $y = \log_b c$. Then, by the definition of logarithms,

$$a^x = b \quad \text{and} \quad b^y = c.$$

This means that

$$c = b^y = (a^x)^y = a^{xy},$$

with the last equality following from the laws of indices. Since $c = a^{xy}$, by the definition of logarithms this means that

$$\log_a c = xy = \log_a b \times \log_b c.$$

$\square$